

## Hofstadter's Cocoon

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**Abstract** Hofstadter showed that the energy levels of electrons on a lattice plotted as a function of magnetic field form an beautiful structure now referred to as “Hofstadter’s butterfly”. We study a non-Hermitian continuation of Hofstadter’s model; as the non-Hermiticity parameter  $g$  increases past a sequence of critical values the eigenvalues successively go complex in a sequence of “double-pitchfork bifurcations” wherein pairs of real eigenvalues degenerate and then become complex conjugate pairs. The associated wavefunctions undergo a spontaneous symmetry breaking transition that we elucidate. Beyond the transition a plot of the real parts of the eigenvalues against magnetic field resembles the Hofstadter butterfly; a plot of the imaginary parts plotted against magnetic fields forms an intricate structure that we call the Hofstadter cocoon. The symmetries of the cocoon are described. Hatano and Nelson have studied a non-Hermitian continuation of the Anderson model of localization that has close parallels to the model studied here. The relationship of our work to that of Hatano and Nelson and to PT transitions studied in PT quantum mechanics is discussed.

**Keywords** PT quantum mechanics · Hofstadter butterfly · Harper’s Equation

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In 1976 Hofstadter demonstrated that the classic Landau problem, describing an electron on a 2-D surface with a normal magnetic field, has a much more interesting eigenvalue spectrum when solved on a lattice than in the continuum [1]. For the lattice, a plot of energy levels as a function of magnetic field forms an intricate and beautiful pattern known as Hofstadter’s butterfly, whereas in the continuum the energy levels disperse linearly with magnetic field, forming the much simpler “Landau fan”. A similar butterfly emerges in the continuum when the Landau levels are weakly perturbed by a periodic potential. Until recently the butterfly had eluded experimental observation due to the unattainably high magnetic fields needed to insert a flux quantum through an atomic scale unit cell. However an electromagnetic analog of Hofstadter’s butterfly was observed using guided microwaves [2]. Very recent experiments have created a Moire superlattice by placing bilayer graphene on a suitable substrate resulting in a potential with a periodicity of hundreds of Angstrom [3],[4], [5]. These experiments have finally observed the butterfly in the original context of electrons in a perpendicular magnetic field.

In this work we show that the Hofstadter model is the Hermitian limit of a more general non-Hermitian Hamiltonian. In the Hermitian limit the spectrum of the Hofstadter model is real. But as the non-Hermiticity parameter  $g$  increases past a sequence of critical values the eigenvalues successively go complex in pairs. As  $g$  passes through a critical value a pair of real eigenvalues degenerates and thereafter becomes a conjugate pair of complex eigenvalues. At the critical value the corresponding eigenfunctions undergo a spontaneous symmetry breaking transition. Our study of the evolution of the spectrum with  $g$  reveals a rich and intricate pattern of symmetry breaking transitions. These transitions can be visualized by plotting the real and imaginary parts of the eigenvalues against the magnetic field. The plots of the real parts resemble the Hofstadter butterfly; we dub the plots of the imaginary parts the Hofstadter cocoon. Apart from its intrinsic interest, the behavior that we calculate is experimentally accessible to microwave experiments [2]

similar to those first used to observe the Hofstadter butterfly. Recently Yuce has considered the same system in the context of coupled optical waveguides [6]; his findings are highly relevant and complementary to the work presented here.

Non-Hermitian continuations of Hermitian models have been fruitfully studied in condensed matter physics since the seminal work of Dyson on spin-waves in a ferromagnet [7]. Our work has a close relationship to that of Hatano and Nelson, who studied a non-Hermitian Hamiltonian that described the classical statistical mechanics of vortex line depinning [8] [9], [10]. The model of Hatano and Nelson may be regarded as the non-Hermitian continuation of the Anderson model of localization [11]. Hatano and Nelson made the interesting discovery that in their model, the transition to complex eigenvalues was accompanied by a delocalization transition in the associated wavefunctions. Shortly thereafter, Bender and co-workers initiated the study of non-Hermitian Hamiltonians with PT symmetry [12] [13]. Within PT quantum mechanics the transition to complex eigenvalues may be regarded as the spontaneous breaking of PT symmetry. Interest in this remarkable new kind of symmetry breaking has been heightened by the observation of such a transition in optical systems with PT symmetry [14], [15]. We show below that the transitions we study as well as the delocalization transition of Hatano and Nelson are analogs of the PT transition inasmuch as they involve the spontaneous breaking of an anti-unitary symmetry.

To provide context for our results we briefly recall the Hofstadter and Hatano/Nelson models. In the continuum, electrons in the  $x$ - $y$  plane subject to a perpendicular magnetic field  $\mathbf{B} = B\hat{\mathbf{z}}$  are governed by the Schrödinger equation

$$-\frac{1}{2} \left[ \frac{\partial^2}{\partial x^2} + \left( \frac{\partial}{\partial y} - ix \right)^2 \right] \psi = E\psi. \quad (1)$$

We have adopted the Landau gauge  $\mathbf{A} = Bx\hat{\mathbf{y}}$  and units wherein  $\hbar = m = eB = 1$ . We take the solution to be separable  $\psi(x, y) = \xi(x) \exp(iky)$ ; then  $\xi$  obeys

$$\left[ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2}(x - k)^2 \right] \xi = E\xi. \quad (2)$$

This is a shifted harmonic oscillator; the energy levels of the system are given by  $E_n = (n + \frac{1}{2})\hbar eB/m$ . Thus we see that the energy levels are evenly spaced with a spacing proportional to the magnetic field, leading to the famous Landau fan. Note that if restrict the electron to a square region of size  $L$  and impose periodic boundary conditions the allowed  $k$  values become quantized but the eigenvalues are not affected [16].

Hofstadter considered electrons on a two dimensional square lattice of lattice constant  $a$  immersed in a uniform perpendicular field [1]. Thus  $x = na$  and  $y = ma$  and the Schrödinger equation takes the form

$$-\tau e^{i2\pi\phi m} \psi_{n+1,m} - \tau e^{-i2\pi\phi m} \psi_{n-1,m} - \tau \psi_{n,m+1} - \tau \psi_{n,m-1} = E\psi_{n,m} \quad (3)$$

where  $\phi = eBa^2/h$  is the magnetic flux per plaquette (in units of  $h/e$ ) and we are working with a linear Landau gauge for the lattice model as well. If we consider a finite system of size  $aL \times aL$  and impose periodic boundary conditions then Dirac's quantization argument requires that the total flux through the sample be an integer or half integer. Periodic boundary conditions are the requirement that the wavefunction is invariant under magnetic translations (rather than ordinary translations) by  $aL$  along the  $n$  and  $m$  directions [17]. In the linear Landau gauge this amounts to the conditions  $\psi_{n+L,m} = \psi_{n,m}e^{-i2\pi\phi Ln}$  and  $\psi_{n,m+L} = \psi_{n,m}$ . Note that for the special case that the flux obeys the condition that  $\phi L$  is an integer the magnetic periodic boundary conditions coincide with ordinary periodic boundary conditions  $\psi_{n+L,m} = \psi_{n,m}$  and  $\psi_{n,m+L} = \psi_{n,m}$ . For simplicity we will restrict attention to these special values of flux for which the solutions to (3) have the separable form

$$\psi_{n,m} = e^{ikn}\xi_m, \quad (4)$$

where  $\xi_m$  obeys Harper's equation,

$$-\xi_{m+1} - \xi_{m-1} - 2\cos(2\pi\phi m + k)\xi_m = E\xi_m, \quad (5)$$

with the boundary condition that  $\xi_{m+L} = \xi_m$ . Periodic boundary conditions also constrain  $k$  to be of the form  $2\pi p/L$  where  $p$  is an integer. The eigenvalues of Harper's equation (4) for all allowed values of  $k$  plotted as a function of  $\phi$  form the famous butterfly. In the infinite size limit the discrete levels of the butterfly blend to form bands with a intricate pattern of gaps. The self-similar structure of these bands was later elucidated by Wannier [18] and MacDonald [19]; Harper's equation also appears in the context of quasicrystals, see for example [20].

Hatano and Nelson studied a one dimensional lattice model governed by

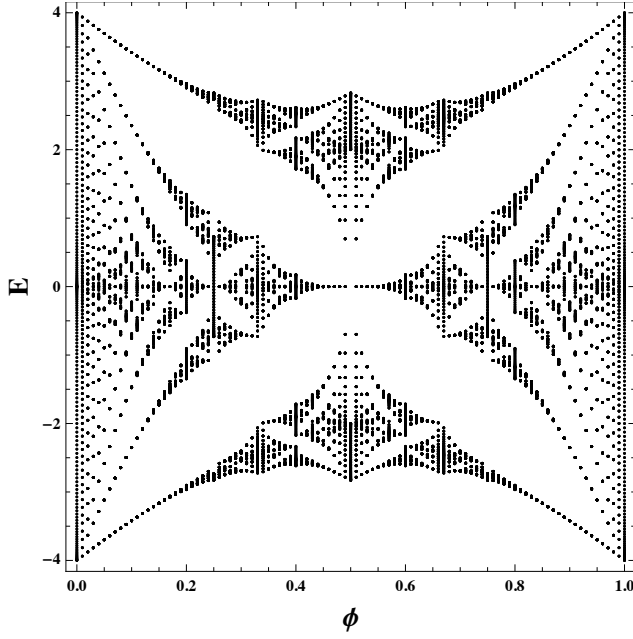
$$-e^g\xi_{m+1} - e^{-g}\xi_{m-1} - V_m\xi_m = E\xi_m \quad (6)$$

with the onsite potentials  $V_m$  taken to be random [8], [9], [10]. Note that in place of phase factors that represent abelian gauge fields on a lattice, Hatano and Nelson inserted real exponential factors in the hopping terms corresponding to an imaginary vector potential. For  $g = 0$  the Hatano and Nelson model coincides with the Anderson model of localization theory [11]; for non-zero  $g$  it corresponds to a non-Hermitian continuation of the Anderson model. Hatano and Nelson found that for small values of  $g$  the eigenvalues remained real but as  $g$  passed a critical value the eigenvalues become complex in conjugate pairs. Moreover the associated eigenfunctions underwent a delocalization transition.

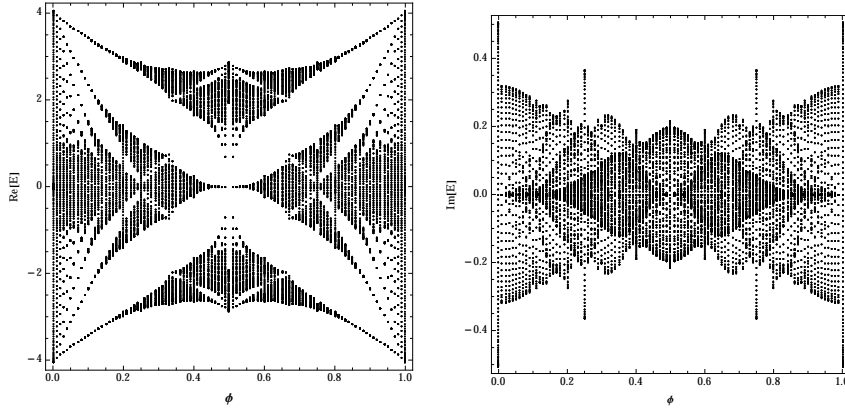
Here we study the Hofstadter model perturbed by an imaginary vector potential; a non-Hermitian continuation of Harper's equation:

$$-e^g\xi_{m+1} - e^{-g}\xi_{m-1} - 2\cos(2\pi\phi m + k)\xi_m = E\xi_m. \quad (7)$$

The exponential factors corresponding to an imaginary vector potential differentiate our model from the Hermitian Harper's equation. Our model differs



**Fig. 1** *Hofstadter's butterfly*. Eigenvalue spectrum of Eq.(7) plotted as a function of flux  $\phi$  for  $g = 0$  and  $L = 200$ .



**Fig. 2** The non-Hermitian butterfly and cocoon. Real (left) and imaginary (right) parts of the eigenvalues of Eq.(7), plotted as a function of flux  $\phi$  for  $g = -0.25$  and  $L = 200$ .

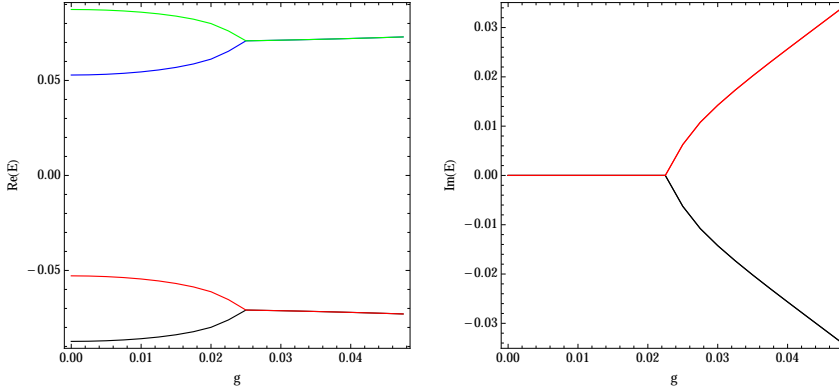
from Hatano and Nelson in that the onsite potential is quasi-periodic rather than random.

Figure 1 shows the Hermitian Hofstadter butterfly: with  $g = 0$  the Hamiltonian in Eq. (7) is Hermitian and the eigenvalues are guaranteed to be real.

Figure 2 depicts the spectrum for  $g = -0.25$ ; . The left panel shows the real parts of the eigenvalues plotted as a function of flux. This plot, which we call the non-Hermitian butterfly, resembles the Hofstadter butterfly qualitatively, although there are small quantitative differences that evolve with  $g$ . The right panel shows the imaginary parts of the eigenvalues plotted as a function of flux, which form a complex structure we dub the Hofstadter cocoon. In Figure 3 we illustrate the double pitchfork morphology of the transition to complex eigenvalues that is characteristic of our model as well as of the model of Hatano and Nelson [10] and of the transitions studied in PT quantum mechanics [13]. Below the critical value of  $g$  we show a pair of eigenvalues that are non-degenerate and real [green and blue curves in Fig 3(a)]. At the critical value of  $g$  the two real eigenvalues degenerate. Above the critical value of  $g$  the two eigenvalues become a complex conjugate pair. Thus their real parts remain degenerate [green curve in Fig 3(a)] but their imaginary parts are equal in magnitude and opposite in sign [black and red curves in Fig 3(b)]. Thus the real and imaginary parts of this pair of eigenvalues plotted as a function of  $g$  form a pair of complementary pitchforks as shown in Fig 3. Thus far the morphology of the transition is the same as in the model of Hatano and Nelson and in PT quantum mechanics. However our transition has another feature by virtue of a symmetry of our model that if  $E$  is an eigenvalue of our model then so is  $-E$  [symmetry (b) discussed in the paragraph below]. Thus it follows that at every transition two pairs of real eigenvalues will degenerate [blue and green curves and red and black curves in Fig 3(a)]. Thereafter these two pairs form two conjugate pairs that remain related by a sign change. As  $g$  is increased further additional quartets of real eigenvalues degenerate and form complementary conjugate pairs (that is, conjugate pairs that differ in sign). An impression of these transitions is conveyed by Fig 4 which depicts the imaginary parts of the eigenvalues as a function of  $g$  for a given value of  $\phi$ . For the particular flux shown in Fig 4 the first transition to complex eigenvalues occurs at a non-zero value of  $g$ . Thus there is a small range of  $g$  values over which all the eigenvalues remain real even though their reality is no longer guaranteed by Hermiticity. For other values of the flux however the first transition to complex eigenvalues happens immediately at  $g = 0$ .

The Hofstadter butterfly has symmetries that Hofstadter identified and proved[1]. The spectra we calculate have a number of analogous symmetries: (a) The spectrum is periodic as a function of the flux  $\phi$  with period 1. (b) If  $E$  is an eigenvalue for a given flux so is  $-E$ . (c) The spectrum is the same for flux  $\phi$  and  $1 - \phi$ . (d) The spectrum is the same for  $g$  and  $-g$ . To prove (b) note that if  $\xi_m$  is a solution to eq (7) with eigenvalue  $E$  and wave-vector  $k$  then  $(-1)^m \xi_m$  with wave-vector  $k + \pi$  is a solution to eq (7) but with eigenvalue  $-E$ . To prove (c) note that if  $\xi_m$  is a solution to eq (7) with wave-vector  $k$  and flux  $\phi$  then it is also a solution with wave-vector  $-k$  and flux  $1 - \phi$  with an unchanged eigenvalue  $E$ . The proofs of (a) and (d) are comparatively simple and are omitted.

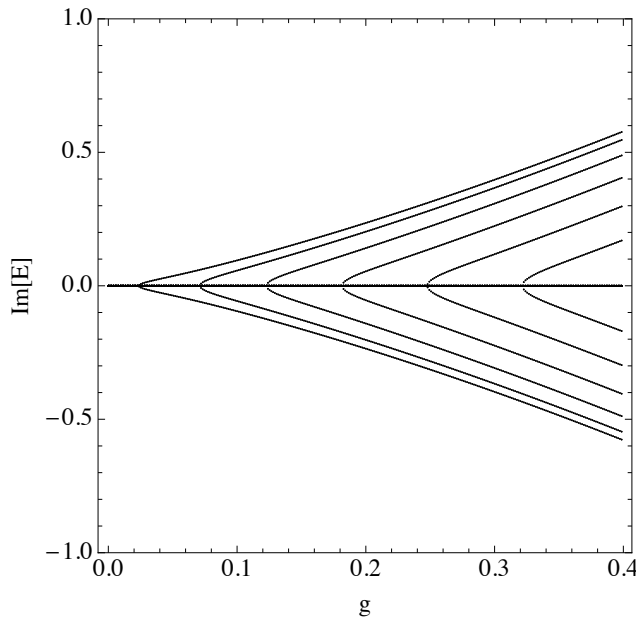
It is not coincidental that the transition to complex eigenvalues has the same double pitchfork form in our model and in that of Hatano and Nelson



**Fig. 3** The double pitchfork bifurcation. Two pairs of real eigenvalues degenerate at a critical value of  $g$ . Thereafter the eigenvalues form two conjugate pairs that are complementary in the sense that they differ in sign. The evolution of the real parts of the four eigenvalues is shown in (a) and the evolution of the imaginary parts in (b). In this plot  $L = 50$  and  $\phi = 0.02$ .

as it has in  $PT$  quantum mechanics. Recall that in  $PT$  quantum mechanics the Hamiltonian must be  $PT$  symmetric *i.e.* it must commute with the anti-linear operator  $PT$ . It follows that if  $\psi$  is an eigenfunction of the Hamiltonian with eigenvalue  $E$  then  $PT\psi$  is an eigenfunction with eigenvalue  $E^*$ . Now if eigenstates of the Hamiltonian can be found that are invariant under  $PT$  the eigenvalues will be real. But if the eigenstates are not invariant under  $PT$  then  $PT$  symmetry is spontaneously broken and the eigenvalues will be complex [13]. The double pitchfork structure emerges from the requirement that when  $PT$  symmetry is broken the eigenvalues must come in conjugate pairs. In the Hatano and Nelson model and in our model, the role of  $PT$  symmetry is played by the anti-linear operation of conjugation. Inspection of eq (6) and eq (7) shows that if  $\xi_m$  is an eigenfunction with eigenvalue  $E$  then  $\xi_m^*$  is an eigenfunction with eigenvalue  $E^*$ . Thus when the eigenvalues become complex they must do so in conjugate pairs explaining the double pitchfork structure observed by us and by Hatano and Nelson. Furthermore the transition to complex eigenvalues is revealed to be a manifestation of a spontaneous breaking of conjugation symmetry. Much effort has been devoted to interpreting delocalization transitions in terms of the concept of spontaneous symmetry breaking [21]. That the delocalization transition of Hatano and Nelson may be regarded as the spontaneous breaking of conjugation symmetry does not appear to have been remarked upon before.

We conclude with some open questions that deserve further study. Design of an electromagnetic analog of our system similar to the one used to realize Hofstadter's butterfly [2] is very desirable as it would make our model amenable to experimental study. The recursive structure of the Hofstadter butterfly was elucidated by Wannier [18] who tracked the evolution of band gaps with flux. We expect the gap trajectories of the non-Hermitian butter-



**Fig. 4** Sequence of transitions to complex eigenvalues. The evolution of the imaginary parts of all the eigenvalues is shown as a function of  $g$  for  $L = 50$  and  $\phi = 0.02$ .

fly to be essentially the same as the Hofstadter butterfly but the recursive structure of the cocoon would be qualitatively different from the butterfly and remains to be elucidated. Thouless et al demonstrated that the bands of the Hofstadter model are characterized by topological integers [22]. The effect of the non-Hermitian perturbation on these topological invariants is worthy of further investigation.

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